Equivalence of the Fleming-Viot and Look-down models of Muller's ratchet

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Abstract

We consider Muller's ratchet Fleming-Viot model with compensatory mutations, which is an infinite system of SDE used to study the accumulation of deleterious mutations in asexual population including mutations and selection. We construct a specific look-down model, and we prove that it is equivalent to the previous Muller's ratchet model.

Keywords : Muller's ratchet, Fleming-Viot, Look-down, Tightness, SDEs.

Introduction

The look-down model was first introduced by Donnelly and Kurtz (see [5] and [7]). The idea is to distribute the population on sites indexed by $i \geq 1$, with exactly one individual per site. In the "modified look-down model" of Donnelly and Kurtz, the population evolves in continuous time as follows : for each pair of sites (i, j), at rate $c > 0$, the individual sitting on site $i \wedge j$ gives birth to an individual sitting on site $i \vee j$, and all individuals sitting on a site greater than or equal to $i \vee j$ are shifted to the right, that is to say each of those individual will move to the site which is at his right.

The two main differences between this model and the Moran model, are that first, the arrows representing births are always pointing to the right, that is to say an individual sitting on site i can only give birth to an individual on a site j with $j > i$. This ensures that the infinite model is well defined, since $\forall n \geq 0$, the evolution of the individuals sitting on the first n sites only depends on births happening on the first n site. The asymmetry which result from this choice is compensated by exchangeability, which is an important property of the look-down model.

The second difference is that the individual who was sitting on the site were the offspring took place does not disappear, but instead is moved to the right, just as all the individuals which are on a site to his right.In [6] Donnelly and Kurtz added selection for a finite number of type of individuals to their model, which involved additional births or possible deaths.

In our model, when a death occurs, the individual who dies is removed from the population, and each of the individuals sitting on a site to the right of his site are shifted to the left. This is a model of death different from the one in [6]. Note that with those deaths, the infinite model is no longer immediately well-defined, since $\forall n \geq 0$, the evolution of the individuals sitting on the first n sites depend, in case of death, on the individual sitting on the following sites. For example, in [3] the authors show that when there are two type of individuals and one death rate, the infinite model is well defined. It has also been shown for several specific models (see e.g. [5], [6], [3]) that the look-down model can be seen as a particle representation for the Fleming-Viot measure-valued diffusion.

In this paper, we consider a look-down version of the Muller's ratchet model with compensatory mutations, which have been suggested by A. Wakolbinger in a personal communication. The model will have mutations in addition of selection, and will involve an infinite number of types of individuals, and an infinite number of selection rates. It is not obvious that in that case the infinite model can be defined, since the death rate is not bounded (see below for the definition of our look-down model). Therefore, we will begin by defining our model in a finite population case, and will show that this model does have a limit (see Theorem 2) when the size of the population tends to infinity.

More precisely, we will consider an asexual population where two types of mutations occur : first, deleterious mutations which have the same value and are independent so they have cumulative effects, and secondly compensatory mutations which cancel deleterious mutations one by one (and thus not having an effect on individual who carry no deleterious mutations). Since the type of one individual is determined by the number of uncanceled deleterious mutations he carries, we will only account those uncanceled mutations when we will speak about carried mutations. We will also suppose that all the mutations are transmitted from any individual to his offsprings. We define a modified look-down model called (L^n), with a finite fixed number n of individuals. Let $\eta_i^n(t)$ be the number of mutations carried by the individual sitting on level i at time t, $1 \leq i \leq n$, and X_k^n the proportion of individuals with k deleterious mutations. We also define $X^n = (X_k^n, k \geq 0)$. The following events occur :

Mutation : Each individual gains one deleterious mutation at rate λ, and mutations are canceled at rate γ.

Selection : $\forall 1 \leq i \leq n$, the individual sitting at site i dies at rate $\alpha\eta_i^n(t)$. When it happens, all individuals sitting on site j with $j > i$ are moved to the left, and we put at the n-th site an individual, whose number of mutations is randomly chosen in such a way that this number equals k with probability $\mu(k) = X_k^n(t^-) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{\eta_i^n(t^-)=k}$.

Birth : For each pair of individual sitting at sites i and j with $i < j$, at rate c, the leftmost one gives birth to a child with the same number of mutation at site j, and for all $j' \geq j$ the individual sitting on site j' is moved one step to the right, and the n-th individual dies.

Similarly, we define the model (L^∞) with an infinite population which follows the same rules as L^n with $n = \infty$, and $X^\infty = (X_k^\infty, k \geq 0)$ the infinite vector of the proportions for the model (L^∞). As said before, we will prove that such model is well defined.

For any initial proportion condition $x = (x_k)_{k\geq 0}$, we construct the initial condition for our look-down model as follows : $\forall 0 \leq k \leq n$, $\eta_k^n(0)$ are i.i.d and $\mathbb{P}(\eta_k^n(0) = \ell) = x_\ell$.

Our aim is to prove that this infinite model is equivalent to a the Fleming Viot model of Muller's ratchet with compensatory mutations, that is to say that the proportions X_k^∞ of our model solve the following infinite SDE system (0.1), with the following notations :

$N > 0$ a parameter;

$X_k(t)$ the proportion of individuals with k deleterious mutations at time t;

λ (resp. γ) is the rate at which deleterious mutations (resp. compensatory mutations) occur;

α is the harmfulness of each single deleterious mutation;

$\{B_{k,\ell}, k > \ell \geq 0\}$ are independent brownian motions, and $B_{k,\ell} = -B_{\ell,k}$;

$M_1 = \sum_{k\in\mathbb{N}} kX_k$ denotes the mean number of mutations in the total population,

$M_\ell = \sum_{k\in\mathbb{N}}(k - M_1)^\ell X_k$ is the ℓ–th centered moment, $\forall \ell \geq 2$.

The Fleming–Viot model for Muller's ratchet with compensatory mutations in continuous time is given by the following infinite set of SDEs

$$\begin{cases} dX_k = [\alpha(M_1 - k)X_k + \lambda(X_{k-1} - X_k) + \gamma(X_{k+1} - X_k)]\,dt + \displaystyle\sum_{\ell\geq 0,\ell\neq k}\sqrt{\frac{X_kX_\ell}{N}}dB_{k,\ell} \\ \qquad = [\alpha(M_1 - k)X_k + \lambda(X_{k-1} - X_k) + \gamma(X_{k+1} - X_k)]\,dt + \sqrt{\frac{X_k(1-X_k)}{N}}dB_k \\ X_k(0) = x_k;\ k \geq 1. \end{cases} \tag{0.1}$$

and for $k=0$,

$$\begin{cases} dX_0 = [\alpha M_1 X_0 - \lambda X_0 + \gamma X_1]\,dt + \displaystyle\sum_{\ell\geq 0, \ell\neq 0} \sqrt{\frac{X_0 X_\ell}{N}} dB_{0,\ell} \\ \qquad = [\alpha M_1 X_0 - \lambda X_0 + \gamma X_1]\,dt + \sqrt{\dfrac{X_0(1-X_0)}{N}} dB_0 \\ X_0(0) = x_0. \end{cases}$$

where $(B_k, k\geq 0)$ are standard brownian motion with $\forall k \neq \ell$

$$\left\langle \int_0^t \sqrt{\frac{X_k(s)(1-X_k(s))}{N}} dB_k(s), \int_0^t \sqrt{\frac{X_\ell(s)(1-X_\ell(s))}{N}} dB_\ell(s) ds \right\rangle = -\int_0^t \frac{X_k(s)X_\ell(s)}{N} ds$$

.

We choose our initial condition $x = (x_k)_{k\geq 0}$ such as $x \in \mathcal{X}$, where

$$\mathcal{X}_\rho = \left\{ (x_k)_{k\geq 0},\ \text{such as } \forall k \geq 0, 0 \leq x_k \leq 1, \sum_{k\geq 0} x_k = 1 \text{ and } \sum_{k\geq 0} x_k e^{\rho k} < \infty \right\}.$$

and

$$\mathcal{X} = \cup_{\rho>0} \mathcal{X}_\rho = \left\{ (x_k)_{k\geq 0},\ \text{such as } \forall k \geq 0, 0 \leq x_k \leq 1, \sum_{k\geq 0} x_k = 1 \text{ and } \exists \rho > 0 \text{ such as } \sum_{k\geq 0} x_k e^{\rho k} < \infty \right\}$$

Note that $\mathcal{X}_\rho$ is complete for the distance $d(x,y) = \sum_{k\geq 0} |x_k - y_k| e^{\rho k}$.

This model is a slight variation of the one proposed by P. Pfaffelhuber, P.R. Staab and A. Wakolbinger in [9]. Indeed in their model, they chose a compensatory mutation rate which which was proportional to the number of carried deleterious mutations i.e. γk for the individuals with k deleterious mutations, $\forall k \geq 0$. The whole following proof can be applied to the their model with very little modifications, but we chose to study our alternative model since in our case, the proof of the convergence to the infinite model (see section 4) is slightly harder, because it involves getting an upper bound on the mean number of deleterious mutations in the population. And in our case, compensatory mutations occur less frequently, then there is more deleterious mutations, and obtaining the upper-bound is slightly more difficult. The set $\mathcal{X}$ is already used in [9] by P. Pfaffelhuber, P.R. Staab and A. Wakolbinger to prove existence and uniqueness of the solution of their Fleming-Viot infinite system of SDE's starting from an initial condition in $\mathcal{X}$. Their proof can be applied to our model (all the necessary reckonings are done through this paper, like e.g. exponential moments) so we have the following Proposition :

Proposition 0.1 *The infinite system of SDE* (0.1) *is well posed, that is to say there is one and only one weak solution* $X = (X_k(t), t \geq 0, k \geq 0)$ *for any given initial value* $x \in \mathcal{X}$.

In the sequel, X will refer to this unique solution.

To reach our objective, we will proceed as follows :

In a first section, we will calculate the generator of the model (L^n), and consider its limit when $n \to \infty$. This will give some hints about the equation solved by the limit of the (X^n), and is used in the proof of existence and unicity in [9] to obtain the corresponding martingale problem.

In the second section, we will establish the tightness of $(X^n, n \geq 0)$ by writing X^n as the solution of an infinite SDE system. Then by calculating the limit of the previous system of SDE, which will require to carefully study M_1^n and to prove that

$$\lim_{n\to\infty} \sum_{k\geq 0} kX_k^n = \sum_{k\geq 0} \lim_{n\to\infty} kX_k^n,$$

we will deduce the first Theorem :

Theorem 1 $\forall k \geq 0$, $(X_k^n, n \geq 0)$ *is tight, and the family of the limits in law is the solution* X *starting from* x *of (0.1).*

In the third section we will use a method inspired from [3] to construct (L^∞) and show that it is well defined. Then in the fourth section we will prove that (L^∞) has the exchangeability property, like said in the following Theorem :

Theorem 2 *The model* L^∞ *is well defined, and is the limit of the* L^n *when* $n \to \infty$ *as follows :* $\forall i > 0$, $\forall t > 0$, $\eta_t^{i,n}$ *converges a.s. and we call* $\eta_t^{i,\infty}$ *its limit. Moreover, it has the exchangeability property, that is to say if the* $(\eta_0^{i,\infty})_{i\geq 1}$ *are exchangeable, then* $\forall t > 0$, *the* $(\eta_t^{i,\infty})_{i\geq 1}$ *are exchangeable. As a consequence,*

$$X^\infty \equiv X \text{ (equality in law).}$$

Finally in the fifth section we will combine the previous results we obtained to improve the obtained convergences. Then we will deduce the third and final Theorem :

Theorem 3 $\forall T \geq 0$, $sup_{0\leq t\leq T} \sum_{k\geq 0} |X_k^n(t) - X_k^\infty(t)| \to 0$ *in probability.*

1 The generator

In this section, we will determine the generator A_n for the process $X = (X^n_k, k \geq 0)$, and consider its limit when $n \to \infty$. This will give some hints about the equation solved by the limit of the (X^n), and can be used to determine the associated martingale problem. The proof of the Theorems will begin the next section.

We define $e^n_k = (\frac{\delta_{\ell,k}}{n}, \ell \geq 0) \in Z^{\mathbb{N}}_+$, and $e_k = ne^n_k$.

For all $f \in \mathcal{C}^{\in}_{\lfloor}(Z^{n+1}_+, \mathbb{R})$,

Mutation : Since there are nX_k individuals carrying k deleterious mutations,

$$A^{mut}_n f(x) = \sum_{k\geq 0} \lambda n x_k \left(f(x - e^n_k + e^n_{k+1}) - f(x) \right) + \sum_{k \geq 1} \gamma n x_k \left(f(x - e^n_k + e^n_{k-1}) - f(x) \right).$$

Selection : Since there are nX_k individuals of type k, and X_ℓ is the probability that the new individual has ℓ deleterious mutations :

$$A^{sel}_n f(x) = \sum_{k,\ell \geq 0, \ell \neq k} n\alpha k x_k x_\ell \left(f(x - e^n_k + e^n_\ell) - f(x) \right).$$

Birth: For each $1 \leq i \leq n$, the individual sitting on level i gives birth at rate $c(n-i)$, while the probability that both he carries k deleterious mutations, and the individual sitting on level n carries ℓ mutations ($\ell \neq k$) is :

$$\begin{aligned}\mathbb{P}(\eta_i = k, \eta_n = \ell) &= \mathbb{P}(\eta_i = k)\mathbb{P}(\eta_n = \ell | \eta_i = k) \\ &= X_k X_\ell \frac{n}{n-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}A^{bir}_n f(x) &= \sum_{k,\ell \geq 0, \ell \neq k} c \frac{n(n-1)}{2} x_k x_\ell \frac{n}{n-1} \left(f(x + e^n_k - e^n_\ell) - f(x) \right). \\ &= \sum_{k,\ell \geq 0, \ell \neq k} c \frac{n^2}{2} x_k x_\ell \left(f(x + e^n_k - e^n_\ell) - f(x) \right).\end{aligned}$$

We obtain the generator for our process :

$$\begin{aligned}
A_n f(x) &= A_n^{mut} + A_n^{sel} + A_n^{bir} \\
&= \sum_{k\geq 0} \lambda n x_k \left(f(x - e_k^n + e_{k+1}^n) - f(x) \right). + \sum_{k\geq 1} \gamma n x_k \left(f(x - e_k^n + e_{k-1}^n) - f(x) \right). \\
&+ \sum_{k,\ell\geq 0,\ell\neq k} n\alpha k x_k x_\ell \left(f(x - e_k^n + e_\ell^n) - f(x) \right) + \sum_{k,\ell\geq 0,\ell\neq k} c\frac{n^2}{2} x_k x_\ell \left(f(x - e_k^n + e_\ell^n) - f(x) \right).
\end{aligned}$$

Now we will estimate its limits when $n \to \infty$.
$\forall x \in \mathcal{X}$, $\exists \rho > 0$ such as $x \in \mathcal{X}_\rho$, and $\forall f \in \mathcal{C}_b^2(\mathcal{X}_\rho, \mathbb{R})$,

$$\begin{aligned}
\lim_{n\to\infty} A_n f(x) &= \lim_{n\to\infty} A_n^{mut} f(x) + \lim_{n\to\infty} A_n^{sel} f(x) + \lim_{n\to\infty} A_n^{bir} f(x) \\
&= (-\lambda x_0 + \gamma x_1)\frac{\partial f}{\partial x_0}(x) + \sum_{k\geq 1} \left(\lambda(x_{k-1} - x_k) + \gamma(x_{k+1} - x_k) \right) \frac{\partial f}{\partial x_k}(x) \\
&+ \alpha \sum_{k\geq 0} (M_1 - k)\, x_k \frac{\partial f}{\partial x_k}(x) + \sum_{k\geq 0} \frac{c}{2} x_k (1 - x_k) \frac{\partial^2 f}{\partial x_k^2}(x) - \frac{c}{2} \sum_{k\geq 0} \sum_{\ell\neq k} x_k x_\ell \frac{\partial^2 f}{\partial x_k \partial x_\ell}(x).
\end{aligned}$$

Note that if we choose $c = \frac{1}{N}$, we obtain the generator corresponding to the SDE system (0.1). We will prove in the following section that the solution of the Muller's ratchet model is indeed the limit in law of X^n.

2 Tightness and weak convergence

In order to prove that the limit of the X^n solves (0.1), we first need the process $X^n(t)$ to converge in some way. We will prove the tightness of $(X_k^n)_{n\in\mathbb{Z}_+}$ $\forall k \in Z_+$ in $D([0, +\infty])$ with the Skorohod metric. Since M_1^n appears in the equations of X^n (see (2.1)), we will aso need to prove the tightness of $(M_1^n)_{n\in\mathbb{Z}_+}$.Note that $\forall k, n \geq 0$ $X_k^n(0) \in [0, 1]$, which implies that $(x^n, n \geq 0)$ is tight.

Proposition 2.1 $\forall k \geq 0$, $(X_k^n, n \geq 0)$ *and* $(M_1^n, n \geq 0)$ *are tight in* $D([0, +\infty[)$.

To prove this, we will prove the tightness on $[0, T]$ $\forall T > 0$. We will establish the system of SDEs which the X_k^n's and M_1^n solve, and prove some estimates regarding the moments of $(X_k^n, k \geq 0)$.

Let $\left\{P_k^1, P_k^2, P_k^{3,\ell}, P_k^{5,\ell}, k, \ell \geq 0\right\}$ be standard Poisson point processes on $\mathbb{R}_+$, which are mutually independent, except that $P_0^2 = 0$. We also define $\forall k, l \geq 0$ $P_\ell^{4,k} = P_k^{3,\ell}$ and $P_\ell^{5,k} = P_k^{6,\ell}$, and for all $n, j \in \mathbb{Z}_+$, $E_j^n = \sum_{k=0}^\infty k^j X_k^n$.

We have :

$$\begin{aligned}
X_k^n(t) &= X_k^n(0) + \frac{1}{n} P_{k-1}^1 \left(\lambda n \int_0^t X_{k-1}^n(s) ds\right) - \frac{1}{n} P_k^1 \left(\lambda n \int_0^t X_k^n(s) ds\right) \\
&+ \frac{1}{n} P_{k+1}^2 \left(\gamma n \int_0^t X_{k+1}^n(s) ds\right) - \frac{1}{n} P_k^2 \left(\gamma n \int_0^t X_k^n(s) ds\right) \\
&+ \frac{1}{n} \sum_{\ell=0, \ell \neq k}^\infty P_k^{3,\ell} \left(\alpha n \ell \int_0^t X_k^n(s) X_\ell^n(s) ds\right) - \frac{1}{n} \sum_{\ell=0, \ell \neq k}^\infty P_k^{4,\ell} \left(\alpha n k \int_0^t X_k^n(s) X_\ell^n(s) ds\right) \\
&+ \frac{1}{n} \sum_{\ell=0, \ell \neq k}^\infty P_k^{5,\ell} \left(c \frac{n^2}{2} \int_0^t X_k^n(s) X_\ell^n(s) ds\right) - \frac{1}{n} \sum_{\ell=0, \ell \neq k}^\infty P_k^{6,\ell} \left(c \frac{n^2}{2} \int_0^t X_k^n(s) X_\ell^n(s) ds\right).
\end{aligned}$$

Note that one can rewrite those equations as follows for $k \geq 1$, and without the term $-\gamma \int_0^t X_k^n(s) ds$ for $k = 0$:

$$\begin{aligned}
X_k^n(t) &= X_k^n(0) + \lambda \int_0^t X_{k-1}^n(s) ds - \lambda \int_0^t X_k^n(s) ds + \gamma \int_0^t X_{k+1}^n(s) ds - \gamma \int_0^t X_k^n(s) ds \\
&+ \alpha \int_0^t X_k^n(s)(M_1^n(s) - k X_k^n(s)) ds - \alpha k \int_0^t X_k^n(s)(1 - X_k^n(s)) ds + \mathcal{M}_t^{n,k} \\
&= X_k^n(0) + \lambda \int_0^t \left(X_{k-1}^n(s) - X_k^n(s)\right) ds + \gamma \int_0^t \left(X_{k+1}^n(s) - X_k^n(s)\right) ds \\
&+ \alpha \int_0^t X_k^n(s)(M_1^n(s) - k) ds + \mathcal{M}_t^{n,k}, \qquad (2.1)
\end{aligned}$$

where $\forall k \geq 0$ $\mathcal{M}_t^{n,k}$ is a martingale such that

$$\begin{aligned}
\left\langle \mathcal{M}^{n,k} \right\rangle_t &= \frac{1}{n} \lambda \int_0^t \left(X_{k-1}^n(s) + X_k^n(s)\right) ds + \frac{1}{n} \gamma \int_0^t \left(X_{k+1}^n(s) + X_k^n(s)\right) ds \\
&+ \frac{1}{n} \alpha \int_0^t X_k^n(s) \left(M_1^n(s) - 2k X_k^n(s) + k\right) ds + c \int_0^t X_k^n(s) \left(1 - X_k^n(s)\right) ds.
\end{aligned}$$

and, $\forall k \neq \ell$,

$$\begin{aligned}\left\langle \mathcal{M}^{n,k}, \mathcal{M}^{n,\ell} \right\rangle_t = & -\frac{1}{n}\mathbb{1}_{|\ell-k|=1}\lambda \int_0^t X^n_{k\wedge\ell}(s)ds - \frac{1}{n}\mathbb{1}_{|\ell-k|=1}\gamma \int_0^t X^n_{k\vee\ell}(s)ds \\ & - \frac{1}{n}\alpha(\ell+k)\int_0^t X^n_k(s)X^n_\ell(s)ds + c\int_0^t X^n_k(s)X^n_\ell(s)ds.\end{aligned}$$

Let us define, except as usual for $k = 0$ where the term $-\gamma X_0$ is absent,

$$\begin{aligned}\phi^n_k(s) &= \lambda\left(X^n_{k-1}(s) - X^n_k(s)\right) + \gamma\left(X^n_{k+1}(s) - X^n_k(s)\right) + \alpha X^n_k(s)(M^n_1(s) - k), \\ \psi^n_k(s) &= \frac{1}{n}\lambda\left(X^n_{k-1}(s) + X^n_k(s)\right) + \frac{1}{n}\gamma\left(X^n_{k+1}(s) + X^n_k(s)\right) + \frac{1}{n}\alpha X^n_k(s)\left(M^n_1(s) - 2kX^n_k(s) + k\right) \\ &\quad + cX^n_k(s)\left(1 - X^n_k(s)\right).\end{aligned}$$

From the relations between our Poisson processes, we deduce the following identities :

$$\begin{aligned}&\sum_{k=1}^n k\frac{1}{n}\left(P^1_{k-1}\left(\lambda n\int_0^t X^n_{k-1}(s)ds\right) - P^1_k\left(\lambda n\int_0^t X^n_k(s)ds\right)\right) \\ &+ \sum_{k=1}^n k\frac{1}{n}\left(P^2_{k+1}\left(\gamma n\int_0^t X^n_{k+1}(s)ds\right) - P^2_k\left(\gamma n\int_0^t X^n_k(s)ds\right)\right) \\ &= \sum_{k=0}^n \frac{1}{n}P^1_k\left(\lambda n\int_0^t X^n_k(s)ds\right) - \sum_{k=1}^n \frac{1}{n}P^2_k\left(\gamma n\int_0^t X^n_k(s)ds\right)\end{aligned}$$

$$\begin{aligned}&\sum_{k=0}^n k\frac{1}{n}\sum_{\ell=0,\ell\neq k}^\infty P^{3,\ell}_k\left(\alpha n\ell\int_0^t X^n_k(s)X^n_\ell(s)ds\right) - \sum_{k=0}^n k\frac{1}{n}\sum_{\ell=0,\ell\neq k}^\infty P^{4,\ell}_k\left(\alpha nk\int_0^t X^n_k(s)X^n_\ell(s)ds\right) \\ &= \sum_{k=0}^n \frac{1}{n}\sum_{\ell=0,\ell\neq k}^\infty (k-\ell)P^{3,\ell}_k\left(\alpha n\ell\int_0^t X^n_k(s)X^n_\ell(s)ds\right)\end{aligned}$$

$$\sum_{k=0}^{n} k\frac{1}{n}\sum_{\ell=0,\ell\neq k}^{\infty} P_k^{5,\ell}\left(c\frac{n^2}{2}\int_0^t X_k^n(s)X_\ell^n(s)ds\right) - \sum_{k=0}^{n} k\frac{1}{n}\sum_{\ell=0,\ell\neq k}^{\infty} P_k^{6,\ell}\left(c\frac{n^2}{2}\int_0^t X_k^n(s)X_\ell^n(s)ds\right)$$
$$= \sum_{k=0}^{n}\frac{1}{n}\sum_{\ell=0,\ell\neq k}^{\infty}(k-\ell)P_k^{5,\ell}\left(c\frac{n^2}{2}\int_0^t X_k^n(s)X_\ell^n(s)ds\right)$$

Now, since $M_1^n(t) = \sum_{k=0}^{\infty} kX_k^n(t)$, we obtain :

$$M_1^n(t) = M_1^n(0) + \lambda t - \gamma\int_0^t (1 - X_0^n(s))ds - \alpha\int_0^t M_2^n(s)ds + \mathcal{M}_t^n$$

where $\mathcal{M}_t^n$ is a martingale, and

$$\begin{aligned}
\langle\mathcal{M}^n\rangle_t &= \frac{1}{n}\lambda t + \frac{1}{n}\gamma\int_0^t (1 - X_0)ds + \frac{1}{n}\alpha\int_0^t E_2^n(s)M_1^n(s)ds - \frac{2}{n}\alpha\int_0^t E_2^n(s)M_1^n(s)ds \\
&\quad + \frac{1}{n}\alpha\int_0^t E_3^n(s)ds + c\int_0^t E_2^n(s)ds - c\int_0^t M_1^n(s)^2 ds \\
&= \frac{1}{n}\left(\lambda t + \gamma\int_0^t (1 - X_0)ds + \alpha\int_0^t E_3^n(s)ds - \alpha\int_0^t E_2^n(s)M_1^n(s)ds\right) - c\int_0^t M_2^n(s)ds
\end{aligned}$$

Like for the equations of X_k^n, we define :

$$\begin{aligned}
\phi^n(s) &= \lambda - \gamma(1 - X_0^n(s)) + \alpha M_2^n(s), \\
\psi^n(s) &= \frac{1}{n}\left(\lambda + \gamma(1 - X_0^n(s)) + \alpha E_3^n(s) - \alpha E_2^n(s)M_1^n(s)d\right) + cM_2^n(s)
\end{aligned}$$

As a preparation for estimating the above quantities, we first establish the

Lemma 2.2 $\forall T > 0$, $\forall k > 0$, $\sup_{n\in\mathbb{Z}_+}\sup_{0\leq t\leq T}\mathbb{E}(E_k^n(t)) < \infty$.

PROOF :

$\forall k \geq 0$, (except for $k = 0$ where $-\gamma X_0^n$ is absent.)

$$\begin{aligned}
\mathbb{E}(X_k^n(s)) &= \mathbb{E}(X_k^n(0)) + \lambda \mathbb{E}\int_0^t \left(X_{k-1}^n(s) - X_k^n(s)\right) ds + \gamma \mathbb{E}\int_0^t \left(X_{k+1}^n(s) - X_k^n(s)\right) ds \\
&+ \alpha \mathbb{E}\int_0^t \left(X_k^n(s) M_1^n(s) - k X_k^n(s)\right) ds - \alpha \mathbb{E}\int_0^t \left(k X_k^n(s)(1 - X_k^n(s))\right) ds \\
&= \mathbb{E}(X_k^n(0)) + \lambda \mathbb{E}\int_0^t \left(X_{k-1}^n(s) - X_k^n(s)\right) ds + \gamma \mathbb{E}\int_0^t \left(X_{k+1}^n(s) - X_k^n(s)\right) ds \\
&+ \alpha \mathbb{E}\int_0^t (M_1^n - k) X_k^n(s) ds.
\end{aligned}$$

Then we will use a slight variation of Lemma 2.5 from [2], and we obtain that, with the notation $\Psi_n(t,\rho) = \mathbb{E}(\sum_{k\leq 0} e^{\rho k} X_k^n)$, $\exists \rho_0 > 0$, $\forall n, t > 0, 0 \leq \rho \leq \rho_0$

$$\Psi_n(t,\rho) \leq \Psi_n(0,\rho) e^{\lambda(e^\rho - 1)t},$$

which is an important inequality since $x_k \in \mathcal{X}$, so $\Psi_\infty(0,\rho) < \infty$, and hence $\sup_{n\geq 0} \Psi_n(0,\rho) < \infty$, and therefore prove the Lemma 2.2.

We recall here the argument (see [2] for more details) :

Let, for $C > 0$

$$\begin{aligned}
\Phi_n(t,\rho) &= \sum_{k\geq 0} X_k^n(t) e^{\rho k}, \\
\Phi_n^C(t,\rho) &= \sum_{k\geq 0} X_k^n(t) (e^{\rho k} \wedge C). \\
\Psi_n^C &= \mathbb{E}\Phi_n^C.
\end{aligned}$$

We deduce from Ito's formula

$$\begin{aligned}
\Psi_n^C(t,\rho) &= \Psi_n^C(0,\rho) + \mathbb{E}\int_0^t \sum_{k\geq 0} \left(\lambda\left(X_{k-1}(r) - X_k(r)\right) + \alpha\left(-k + \sum_{j\geq 0} j X_j(r)\right) X_k(r)\right) \left(e^{\rho k} \wedge C\right) dr \\
&+ \mathbb{E}\int_0^t \left(\gamma X_1 (1 \wedge C) + \sum_{k\geq 1} \gamma\left(X_{k+1}(r) - X_k(r)\right)\left(e^{\rho k} \wedge C\right)\right) dr \\
&\leq \Psi_n^C(0,\rho) + \mathbb{E}\int_0^t \left(\lambda\left(e^\rho \Phi_n^C(r) - \Phi_n^C(r)\right) - \alpha \sum_{k\geq 0} k X_k(r)\left(e^{\rho k} \wedge C\right) + \alpha \sum_{j\geq 0} j X_j(r) \Phi_n^C(r)\right) dr,
\end{aligned}$$

because we work with $\rho > 0$, so $Ce^{-\rho} \leq C$ and

$$\gamma X_1(1 \wedge C) + \sum_{k\geq 1} \gamma \left(X_{k-1}(r) - X_k(r)\right)\left(e^{\rho k} \wedge C\right) = \gamma \sum_{k\geq 1} X_k \left(e^{\rho(k-1)} \wedge C - e^{\rho k} \wedge C\right) \leq 0.$$

Moreover, we have (see Corollary 2.4 in [2]) :

$$\sum_{j\geq 0} jX_j^n(r)\,\Phi_n^C(r) - \sum_{j\geq 0} j\left(e^{\rho j} \wedge C\right) X_j^n \leq 0,$$

and since our functions are bounded, we can invert $\mathbb{E}$ and $\int$,

$$\Psi_n^C(t,\rho) \leq \Psi_n^C(0,\rho) + \int_0^t \left(\lambda\left(e^{\rho} - 1\right)\right)\Psi_n^C(r,\rho)\,dr.$$

The result is a consequence of the Gronwall inequality, and the monotone convergence Theorem. ♢

Proof of Proposition 2.1 : Now we take a $T > 0$. Lemma 2.2 implies that $\exists c_1 > 0$ such as $\sup_{n\in\mathbb{Z}_+}\sup_{0\leq t\leq T}\mathbb{E}(M_1^n(t)) < c_1$. Hence,

$$\begin{aligned}
\sup_{n\in\mathbb{Z}_+}\sup_{0\leq t\leq T}\mathbb{E}(|\phi_k^n(s)|) &\leq \lambda + \gamma + \alpha\left(\sup_{n\in\mathbb{Z}_+}\sup_{0\leq t\leq T}\mathbb{E}(M_1^n(s)) \vee k\right), \\
&\leq \lambda + \gamma + \alpha\left(c_1 \vee k\right), \\
\sup_{n\in\mathbb{Z}_+}\sup_{0\leq t\leq T}\mathbb{E}(|\psi_k^n(s)|) &\leq c + \frac{2}{n}(\lambda+\gamma) + \frac{1}{n}\alpha(k + c_1 \vee k).
\end{aligned}$$

And, for any family of stopping time $(\tau_n)_{n\geq 0}$, $\forall \eta > 0$, $\forall \varepsilon > 0$, if we choose $\theta = \frac{\varepsilon\eta}{\lambda+\gamma+\alpha(c_1\vee k)}$,

$$\begin{aligned}
\sup_{n\in\mathbb{Z}_+}\sup_{\delta\leq\theta}\mathbb{P}(|\int_{\tau_n}^{\tau_n+\delta}\phi_k^n(s)ds| \geq \eta) &\leq \sup_{n\in\mathbb{Z}_+}\sup_{\delta\leq\theta}\mathbb{P}(\int_{\tau_n}^{\tau_n+\delta}|\phi_k^n(s)|ds) \geq \eta) \\
&\leq \sup_{n\in\mathbb{Z}_+}\sup_{\delta\leq\theta}\frac{\theta}{\eta}\sup_{0\leq s\leq T}\mathbb{E}(|\phi_k^n(s)|)) \\
&\leq \frac{\theta}{\eta}\left(\lambda + \gamma + \alpha\left(c_1 \vee k\right)\right) \leq \varepsilon.
\end{aligned}$$

Likewise, for ψ_k^n, by choosing $\theta = \frac{\varepsilon\eta}{c+\frac{2}{n}(\lambda+\gamma)+\frac{1}{n}\alpha(k+c_1\vee k)}$,

$$\sup_{n\in\mathbb{Z}_+}\sup_{\delta\le\theta}\mathbb{P}(|\int_{\tau_n}^{\tau_n+\delta}\psi_k^n(s)ds|)\ge\eta)\le\frac{\theta}{\eta}\left(c+\frac{2}{n}(\lambda+\gamma)+\frac{1}{n}\alpha(k+c_1\vee k)\right)\le\varepsilon.$$

The bounded variation term satisfies Aldous' tighness criterion. Since $\left\langle\mathcal{M}^{n,k}\right\rangle_t$ satisfies the criterion as well, so does $\mathcal{M}^{n,k}$ by Rebolledo's result, then X_k^n is tight (see [8]).

Similarly, from Lemma 2.2, $\exists c_2\ge c_1$ such as

$$\sup_{n\in\mathbb{Z}_+}\sup_{0\le t\le T}\mathbb{E}(M_1^n(t))<c_2,$$
$$\sup_{n\in\mathbb{Z}_+}\sup_{0\le t\le T}\mathbb{E}(M_2^n(t))\le\sup_{n\in\mathbb{Z}_+}\sup_{0\le t\le T}\mathbb{E}(E_2^n(t))<c_2,$$
$$\sup_{n\in\mathbb{Z}_+}\sup_{0\le t\le T}\mathbb{E}(E_3^n(t))<c_2.$$

Hence for M_1^n we have :

$$\sup_{n\in\mathbb{Z}_+}\sup_{0\le t\le T}\mathbb{E}(|\phi^n(s)|)\le\lambda+\gamma+\alpha c_2,$$
$$\sup_{n\in\mathbb{Z}_+}\sup_{0\le t\le T}\mathbb{E}(|\psi^n(s)|)\le\frac{1}{n}\left(\lambda+\gamma+\alpha(c_2+c_2^2)\right)+2c_2,$$

so by choosing $\theta=\frac{\varepsilon\eta}{\lambda+\gamma+\alpha c_2}$ for phi^n and $\theta=\frac{\varepsilon\eta}{\frac{1}{n}\left(\lambda+\gamma+\alpha(c_2+c_2^2)\right)+2cc_2}$ for ψ_n, we can hold the same reasoning and use Aldous' tighness criterion. Hence the result. $\diamondsuit$

Now we can proceed with the

PROOF OF THEOREM 1 : From Proposition 2.1, we consider a strictly increasing sequence $(n_\ell)_{\ell\in\mathbb{N}}$ of integers, constructed by the diagonal extraction procedure, such that $\forall k\ge 0$, the family $\left(M_1^{n_\ell},X_j^{n_\ell},0\le j\le k\right)$ converges weakly, for the Skorohod topology of $D([0,\infty],\mathbb{R}^{k+2})$ when $\ell\to\infty$, and we call $\left(M_1',X_j',0\le j\le k\right)$ its limit.

We will continue to write (X_k^n) for $(X_k^{n_\ell})$ and M_1^n for $M_1^{n_\ell}$ to ease the notations.

In order to prove that the limit solves the Muller's ratchet Fleming-Viot system of SDEs (0.1), we first need to prove that $M_1'=\sum_{k\ge0}kX_k'$.

Since we know that $(M_1^n(t),n\ge0)$ is tight in $D([0,T])$, all we need to prove is that

$$\forall f\in C_b([0,T],\mathbb{R}_+),\mathbb{E}\left(\int_0^T f(t)M_1^n(t)dt\right)\to_{n\to\infty}\mathbb{E}\left(\int_0^T f(t)M_1(t)dt\right).$$

Exploiting Lemma 2.2, $\forall\varepsilon>0$, we can choose $K=K_\varepsilon>0$ such that

$$K \geq \frac{\varepsilon}{\|f\|_\infty} \sup_{n\geq 1} \sup_{t\in[0,T]} \mathbb{E}\left(E_2^n(t)\right)$$

,

$$\mathbb{E}\left(\int_0^T f(t) M_1^n(t)dt\right) = \mathbb{E}\left(\int_0^T f(t) \sum_{k=0}^{K} kX_k^n(t)dt\right) + \mathbb{E}\left(\int_0^T f(t) \sum_{k=K+1}^{\infty} kX_k^n(t)dt\right).$$

The first term in the previous right-hand side tends to $\mathbb{E}\left(\int_0^T f(t)\sum_{k=0}^K kX_k(t)dt\right)$ since $\forall k\geq 0$, $X_k^n \Rightarrow X_k$ and only a finite number of them appear.

As for the second term :

$$\begin{aligned}
\mathbb{E}\left(\int_0^T f(t) \sum_{k=K+1}^{\infty} kX_k^n(t)dt\right) &\leq \frac{1}{K+1}\mathbb{E}\left(\int_0^T f(t)E_2^n(t)\right)\\
&\leq \varepsilon,\\
\mathbb{E}\left(\int_0^T f(t) \sum_{k=K+1}^{\infty} kX_k(t)dt\right) &\leq \varepsilon.
\end{aligned}$$

Now, $\forall k\neq\ell$, we can take the joined limit of (X_k^n, X_ℓ^n) noted (X'_k, X'_ℓ), and from the equation (2.1) we obtain that :

$$\begin{cases}
X'_k(t) = X'_k(0) + \lambda\displaystyle\int_0^t \left(X'_{k-1}(s) - X'_k(s)\right) ds + \gamma \int_0^t \left(X'_{k+1}(s) - X'_k(s)\right) ds\\
\qquad\quad + \alpha\displaystyle\int_0^t X'_k(s)(M'_1(s)-k)ds + \mathcal{M}'^k_t,\\
X'_\ell(t) = X'_\ell(0) + \lambda\displaystyle\int_0^t \left(X'_{\ell-1}(s) - X'_\ell(s)\right) ds + \gamma \int_0^t \left(X'_{\ell+1}(s) - X'_\ell(s)\right) ds\\
\qquad\quad + \alpha\displaystyle\int_0^t X'_\ell(s)(M'_1(s)-\ell)ds + \mathcal{M}'^\ell_t,
\end{cases}$$

where

$$\left\langle\left\langle\begin{pmatrix}\mathcal{M}'^k\\ \mathcal{M}'^\ell\end{pmatrix}\right\rangle\right\rangle_t = c\begin{pmatrix}\int_0^t X'_k(s)\,(1-X'_k(s))\,ds & -\int_0^t X'_k(s)X'_\ell(s)\\ -\int_0^t X'_k(s)X'_\ell(s) & \int_0^t X'_\ell(s)\,(1-X'_\ell(s))\,ds\end{pmatrix}$$

From this we deduce that X' satisfies (0.1), hence from the uniqueness in Proposition 0.1 we deduce Theorem 1.

3 Infinite look-down model

In the previous sections, we studied the convergence and the limit of X^n. In this third part, we will study η^n, and show that one can define a look-down model similar to the (L^∞) with an infinite population, and that our truncated system converges towards it as $n \to \infty$. This will be the proof of Proposition 3.2, which is the first part of Theorem 2.

This section has been inspired by [3].

Let us define $\xi_t^{i,n}$ as follows : $\xi_0^{i,n} = i$, and $\forall t > 0$, whenever there is a birth at time t in (L^n) on a level smaller than or equal to $\xi_{t^-}^{i,n}$, we have $\xi_t^{i,n} = \xi_{t^-}^{i,n} + 1$; whenever there is a death at time t in (L^n) on a level smaller than or equal to $\xi_{t^-}^{i,n}$, we have $\xi_t^{i,n} = \xi_{t^-}^{i,n} - 1$. $\xi_s^{i,n} = i$; so $\xi_t^{i,n}$ denotes the level on which the individual who was sitting on level i at time 0 is at time t, with the convention that when this individual is killed, we follow his left neighbor.

We will write $\frac{n}{2}$ instead of $\lfloor \frac{n}{2} \rfloor$ to ease the notations.

We will prove the following result, with p_n as defined below (see (3.3)).

Proposition 3.1 $\forall n \geq 64\alpha(M_1^{2n}(0) + 5\sqrt{n})$,

$$\mathbb{P}\left(\exists 1 \leq i \leq \frac{n}{2}, 0 \leq t \leq T \textit{ such that } \eta_i^n(t) \neq \eta_i^{2n}(t)\right) \leq n\left(\frac{16\alpha n(M_1^{2n}(0)+5\sqrt{n})}{cn^2}\right)^{\frac{n}{2}} + p_{2n}.$$

This can be seen as follows :for n large enough, with a probability which tends to 1, the $n/2$ first individuals only depends on the n first individuals regarding their evolution. With this idea, we will prove the following Proposition which define the infinite model as the limit of the L^n :

Proposition 3.2 *The model L^∞ is well defined, and is the limit of the L^n when $n \to \infty$ as follows : $\forall i > 0$, $\forall t > 0$, $\eta_t^{i,n}$ converges a.s. and we call $\eta_t^{i,\infty}$ its limit.*

PROOF OF PROPOSITION 3.1 : This will be a three steps proof. In the first step, we will couple our process with a birth and death process. Then, in the second step, we will get some estimate for the rate of death, which will give us the Proposition 3.3. Finally, in a third step we will combine the previous sections to prove the Proposition 3.1.

FIRST STEP : Note that

$$\begin{aligned}&\left\{\exists 1 \leq i \leq \frac{n}{2}, 0 \leq t \leq T \text{ such that } \eta_i^n(t) \neq \eta_i^{2n}(t)\right\}\\ &\subset \left\{\exists 1 \leq i \leq n+1, 0 \leq s < t \leq T \text{ such that } \xi_s^{i,2n} > n,\, \xi_t^{i,2n} = \frac{n}{2}\right\}\end{aligned}$$

Indeed, in order to have the first property we need that at least one individual from the (L^{2n}) model reaches the level $n+1$, then the level $\frac{n}{2}$, hence the inclusion.

Let us chose $1 \leq i_0 \leq n$, $0 \leq s_0 < t$ such that $\xi_{s_0}^{i_0,2n} > n$, $\xi_t^{i_0,2n} = \frac{n}{2}$. The rate $\upsilon_1^n(t)$ at which $\xi_t^{i_0,2n}$ decreases at time t due to deaths is such that

$$\begin{aligned}\upsilon_1^n(s) &\leq \sum_{k=1}^{2n} \alpha \eta_k^{2n}(s) = \sum_{k=0}^{\infty} \alpha 2nk X_k^{2n}(s)\\ &\leq \alpha 2n M_1^{2n}(s)\end{aligned}$$

Moreover, the rate $\upsilon_2^n(s)$ at which $\xi_t^{i_0,2n}$ increases after it has reached n and before it reached $\frac{n}{2}$ for $t \geq s_0$ is greater than or equal to $\frac{cn(n-3)}{8}$.

SECOND STEP : Now we need some estimate of $M_1^n(t)$ (in fact we need those estimate for M_t^{2n}, but we work with M_t^n instead to ease the notation, since the inequality still holds, see after the Proposition 3.3). We will use a similar reasoning as in Lemma 3.2 from [2]. Note that we have : $\forall t > 0$, $\forall 0 \leq r \leq T-t$,

$$\begin{aligned}M_1^n(t+r) &\leq M_1^n(t) + \lambda r - \alpha \int_t^{t+r} M_2^n(s)ds + \mathcal{M}_{t+r}^n - \mathcal{M}_t^n\\ &\leq M_1^n(t) + \lambda r + \frac{1}{2n}\left(\lambda r + \gamma r + \alpha \int_t^{t+r} E_3^n(s)ds\right) + \sum_{i=1}^{4} \mathcal{Z}_i^n(t,r),\end{aligned}$$

where, with B_t^1, B_t^2, B_t^3 and B_t^4 four different Brownian motions,

$$\begin{aligned}
\mathcal{Z}_1(t,r) &= \sqrt{\frac{\lambda}{n}}\int_t^{t+r} dB_s^1 - \frac{\lambda r}{2n}\\
\mathcal{Z}_2(t,r) &= \sqrt{\frac{\alpha}{n}}\int_t^{t+r} \sqrt{E_3^n(s)}dB_s^2 - \frac{\alpha}{2n}\int_t^{t+r} E_3^n(s)ds\\
\mathcal{Z}_3(t,r) &= c\int_t^{t+r} \sqrt{M_2^n}dB_s^3 - \alpha\int_t^{t+r} M_2^n ds\\
\mathcal{Z}_4(t,r) &= \sqrt{\frac{\gamma}{n}}\int_t^{t+r} \sqrt{1-X_0^n(s)}dB_s^4 - \frac{\gamma}{2n}\int_t^{t+r} \left(1-X_0^n(s)\right)ds.
\end{aligned}$$

We note that $\exp\left(\mathcal{Z}_1^n\right)$, $\exp\left(\mathcal{Z}_2^n\right)$, $\exp\left(\mathcal{Z}_4^n\right)$ and $\exp\left(2\frac{\alpha}{c^2}\mathcal{Z}_3^n\right)$ are both local martingales and super-martingales. Hence, like in [2], one can easily deduce that $\forall C>0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le r\le T-t}\mathcal{Z}_i^n(t,r)\ge C\right)\le \exp(-C),\ i=1,\ 2 \text{ or } 4\\
&\mathbb{P}\left(\sup_{0\le r\le T-t}\mathcal{Z}_3^n(t,r)\ge C\right)\le \exp(-2\frac{\alpha}{c^2}C).
\end{aligned}$$

Hence,

$$\mathbb{P}\left(\sup_{0\le r\le T-t}\mathcal{Z}_1^n(t,r)+\mathcal{Z}_2^n(t,r)+\mathcal{Z}_3^n(t,r)+\mathcal{Z}_4^n(t,r)\ge 4C\right)\le \exp(-2\frac{\alpha}{c^2}C)+3\exp(-C). \tag{3.1}$$

On the other hand, a consequence of Lemma 2.2 is that $\exists c_3(\lambda,\gamma,\delta),>0$ (since $E_3^6\le E_{18}$) such that

$$\sup_{n\in\mathbb{Z}_+}\sup_{0\le s\le T}\mathbb{E}\left(\left(\lambda+\frac{1}{2n}\left(\lambda+\gamma+\alpha E_3^n(s)\right)\right)^6\right)\le c_3(\lambda,\gamma,\delta).$$

Then,

$$\begin{aligned}
\mathbb{P}\left(\int_t^{t+r}\lambda+\frac{1}{2n}\left(\lambda+\gamma+\alpha E_3^n(s)\right)ds>C\right) &= \mathbb{P}\left(\left(\int_t^{t+r}\lambda+\frac{1}{2n}\left(\lambda+\gamma+\alpha E_3^n(s)\right)ds\right)^6>C^6\right)\\
&\le \frac{r^7}{C^6}\left(c_3(\lambda,\gamma,\delta)\right).
\end{aligned} \tag{3.2}$$

Finally, by using (3.1) and (3.2), we finally obtain the following Lemma :

Lemma 3.3 $\forall n > 0$, $\forall 0 \le t \le T$, $\forall C > 0$,

$$\mathbb{P}\left(\sup_{0\le r\le T-t} M_1^n(t+r) - M_1^n(t) \ge 5C\right) \le \exp(-2\frac{\alpha}{c^2}C) + 3\exp(-C) + \frac{T^7}{C^6}\left(c_3(\lambda,\gamma,\delta)\right).$$

Note that, as announced before, the right member of the inequality is a decreasing function of n, so it is also true for M_t^{2n}. Also, if we take $C = \sqrt{n}$, the quantity

$$p_n = \exp(-2\frac{\alpha}{c^2}\sqrt{n}) + 3\exp(-\sqrt{n}) + \frac{T^7}{n^3}\left(c_3(\lambda,\gamma,\delta)\right) \tag{3.3}$$

is such that $\sum_{n\ge 0} p_n < \infty$.

Third step : Let ρ_t^n (resp. $\tilde{\rho}_t^n$) be a birth and death process, starting from $\rho_0^n = n$ (resp. $\tilde{\rho}_0^n = n$), with a birth rate $\upsilon_2^n(t)$ (resp $\frac{cn(n-2)}{8}$), and a death rate $\upsilon_1^n(t)$ (resp. $\alpha n(M_1^{2n}(0) + 5\sqrt{n})$). Let $\tau^n = \inf\left\{t > 0, \tilde{\rho}_t^n = \frac{n}{2}\right\}$ and $\tau'_{i,n} = \inf\left\{t > 0, \xi_t^{i,2n} = n\right\}$. Then

$$\begin{aligned}
&\mathbb{P}\left\{\exists 1 \le i \le n, 0 \le s_0 \le T, 0 \le t \le T \text{ such that } \xi_{s_0}^{i,2n} = n, \inf_{s_0\le s\le t} \xi_s^{i,2n} = n/2\right\}\\
&\le \mathbb{P}\left\{\exists 1 \le i \le n, \tau'_{i,n} < T, \inf_{0\le s\le T-\tau'_{i,n}} \xi_s^{2n,2n} = n/2\right\}\\
&\le \mathbb{P}\left\{\exists 1 \le i \le n, \tau'_{i,n} < T, \inf_{0\le s\le T-\tau'_{i,n}} \rho_s^n = n/2\right\}\\
&\le \mathbb{P}\left\{\inf_{0\le s\le T} \rho_s^n = n/2\right\}\\
&\le \mathbb{P}\left\{\inf_{0\le s\le T} \rho_s^n = n/2\right\} \bigcap \left\{\sup_{0\le r\le T} M_1^{2n}(r) - M_1^{2n}(0) \le 5\sqrt{n}\right\} + p_{2n}\\
&\le \mathbb{P}\left\{, \exists 0 \le s_0 \le T, \tilde{\rho}_{s_0}^n = \frac{n}{2}\right\} + p_{2n}\\
&\le \mathbb{P}\left\{\tau^n < T\right\} + p_{2n}\\
&\le \mathbb{P}\left\{\tau^n < \infty\right\} + p_{2n}.
\end{aligned}$$

The rest of this proof is an adaptation from Lemma 1.2 in [3]. We now work with n great enough to have $n \ge 64\alpha(M_1^{2n}(0) + 5\sqrt{n})$. Let $(A_k)_{k\ge 1}$ and $(B_k)_{k\ge 1}$ be two mutually independent sequences of i.i.d. exponential random variables, the A_k having $\frac{cn(n-2)}{8}$ for parameter, and the B_k $\alpha n(M_1^{2n}(0) + 5\sqrt{n})$. We have

$$
\begin{aligned}
\mathbb{P}(\tau^n < \infty) &\leq \sum_{k=0}^{\infty} \mathbb{P}(A_1 + ... + A_k > B_1 + ... + B_{k+\frac{n}{2}}) \\
&\leq \sum_{k=0}^{\infty} \mathbb{P}\left(\exp\left(\frac{cn(n-2)}{8}\left(A_1 + ... + A_k - B_1 + ... - B_{k+\frac{n}{2}}\right)\right) > 1\right) \\
&\leq \sum_{k=0}^{\infty} \left(\mathbb{E}\left(\exp\left(\frac{cn(n-2)}{16}A_1\right)\right)\right)^k \left(\mathbb{E}\left(\exp\left(-\frac{cn(n-2)}{16}B_1\right)\right)\right)^{k+\frac{n}{2}} \\
&= \sum_{k=0}^{\infty} 2^k \left(\frac{\alpha n(M_1^{2n}(0) + 5\sqrt{n})}{\alpha n(M_1^{2n}(0) + 5\sqrt{n}) + \frac{cn(n-2)}{16}}\right)^{k+\frac{n}{2}} \\
&\leq \sum_{k=0}^{\infty} \left(\frac{32\alpha n(M_1^{2n}(0) + 5\sqrt{n})}{cn^2}\right)^k \left(\frac{16\alpha n(M_1^{2n}(0) + 5\sqrt{n})}{cn^2}\right)^{\frac{n}{2}} \\
&\leq \left(\frac{16\alpha n(M_1^{2n}(0) + 5\sqrt{n})}{cn^2}\right)^{\frac{n}{2}}
\end{aligned}
$$

$\diamondsuit$

Proof of Proposition 3.2 : Since

$$
\sum_{n\geq 0}\left(n\left(\frac{16\alpha n(M_1^{2n}(0) + 5\sqrt{n})}{cn^2}\right)^{\frac{n}{2}} + p_{2n}\right) < \infty,
$$

it follows from the Borel Cantelli Lemma that

$$
\mathbb{P}\left(\exists N_0, \forall n \geq N_0, \forall 1 \leq i \leq \frac{n}{2}, 0 \leq t \leq T,\, \eta_i^n(t) = \eta_i^{2n}(t)\right) = 1
$$

$\diamondsuit$

4 Exchangeability

In this section, inspired from [3] as well, we will show that this look-down model preserves the exchangeability property, according to the following Proposition, with $\eta = \eta^\infty$:

Proposition 4.1 *If $(\eta_0(i))_{i\geq 1}$ are exchangeable random variables, then $\forall t > 0$, $(\eta_t(i))_{i\geq 1}$ are exchangeable.*

The Proposition will follow from the four following lemmata :

Lemma 4.2 *For any stopping time τ, any $\mathbb{N}$ valued $\mathcal{F}_\tau$-measurable random variable $\mathcal{X}$, if the random vector $\eta_\tau^{\mathcal{X}} = (\eta_\tau(1), ..., \eta_\tau(\mathcal{X}))$ is exchangeable, and τ' is the first time after τ' of an arrow pointing to a level $\leq \mathcal{X}$, a death or a mutation at a level $\leq \mathcal{X}$, then conditionally upon the fact that τ' is the time of a birth, the random vector $\eta_{\tau'}^{\mathcal{X}+1} = (\eta_{\tau'}(1), ..., \eta_{\tau'}(\mathcal{X}+1))$ is exchangeable.*

Proof : To ease the notation we will condition upon $\mathcal{X} = n$ and $\tau' = t$, and denote by $\mathcal{P}$ the associated conditional probability. Let a^{n+1} be a $n+1$ dimensional vector, and for $1 \leq i < j \leq n$

$$A_t^{i,j} = \{ \text{ The birth which occurs at time t involves the pair (i,j) } \}.$$

$\forall \pi \in S_{n+1}$, i.e. π is a permutation of the set $\{1, 2, ..., n+1\}$

$$\begin{aligned}\mathcal{P}(\pi(\eta_t^{n+1}) = a^{n+1}) &= \sum_{1\leq i<j\leq n} \mathcal{P}(\eta_t^{n+1} = \pi^{-1}(a^{n+1}), A_t^{i,j}) \\ &= \sum_{1\leq i<j\leq n} \mathcal{P}(\eta_t(1) = a_1^\pi, ..., \eta_t(n+1) = a_{n+1}^\pi, A_t^{i,j})\end{aligned}$$

By definition of $A_t^{i,j}$,

$$A_t^{i,j} \cap \left\{\eta_t^{n+1} = (a_1^\pi, ..., a_{n+1}^\pi)\right\} \subset \left\{a_i^\pi = a_j^\pi\right\},$$

hence, defining the projection ρ_j:$\mathbb{N}^{n+1} \to \mathbb{N}^n$

$$\rho_j(b_1, ...b_{n+1}) = (b_1, ..., b_{j-1}, b_{j+1}, ...b_{n+1}),$$

we obtain

$$\begin{aligned}
\mathcal{P}(\pi(\eta_t^{n+1}) = a^{n+1}) &= \sum_{1\le i<j\le n} \mathbb{1}_{a_i^\pi = a_j^\pi} \mathcal{P}(\eta_{t^-}^n = \rho_j(\pi^{-1}(a^{n+1})), A_t^{i,j}) \\
&= \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \mathbb{1}_{a_i^\pi = a_j^\pi} \mathcal{P}(\eta_{t^-}^n = \rho_j(\pi^{-1}(a^{n+1}))) \\
&= \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \mathbb{1}_{a_i^\pi = a_j^\pi} \mathcal{P}(\eta_{t^-}^n = \rho_j(a^{n+1}))
\end{aligned}$$

where the second line is obtained by independence of $A_t^{i,j}$ and $\left\{\eta_{t^-}^n = \rho_j(\pi^{-1}(a^{n+1}))\right\}$, and the last one is a consequence of the exchangeability of $\eta_{t^-}^{n+1}$. The result follows. $\diamondsuit$

Lemma 4.3 *For any stopping time τ, any $\mathbb{N}$ valued $\mathcal{F}_\tau$-measurable random variable $\mathcal{X}$, if the random vector $\eta_\tau^{\mathcal{X}} = (\eta_\tau(1), ..., \eta_\tau(\mathcal{X}))$ is exchangeable, and τ' is the first time after τ of an arrow pointing to a level $\le \mathcal{X}$, a death or a mutation at a level $\le \mathcal{X}$, then conditionally upon the fact that τ' is the time of a deleterious mutation, the random vector $\eta_{\tau'}^{\mathcal{X}} = (\eta_{\tau'}(1), ..., \eta_{\tau'}(\mathcal{X}))$ is exchangeable.*

Proof : To ease the notation we will condition upon $\mathcal{X} = n$ and $\tau' = t$, and denote by $\mathcal{P}$ the associated conditional probability. Let a^n be a n dimensional vector, and for $1 \le j \le n$

$B_t^j = \{$ The deleterious mutation which occurs at time t involves the individual sitting on site j) $\}$.

$\forall \pi \in S_n$,

$$\begin{aligned}
&\mathcal{P}(\pi(\eta_t^n) = a^n) = \sum_{1\le j\le n} \mathcal{P}(\eta_t^n = \pi^{-1}(a^n), B_t^j) \\
&= \sum_{1\le j\le n} \mathcal{P}(\eta_t(1) = a_1^\pi, ..., \eta_t(n) = a_n^\pi, B_t^j) \\
&= \sum_{1\le j\le n} \mathcal{P}(\eta_{t^-}(1) = a_1^\pi, ..., \eta_{t^-}(j-1) = a_{j-1}^\pi, \eta_{t^-}(j) = a_j^\pi - 1, \eta_{t^-}(j+1) = a_{j+1}^\pi, ...\eta_{t^-}(n) = a_n^\pi, B_t^j) \\
&= \sum_{1\le j\le n} \frac{\mathbb{1}_{a_j^\pi \ge 1}}{1 + \sum_{\ell=1,\ell\ne j}^n \mathbb{1}_{a_\ell^\pi \ge 1}} \mathcal{P}(\eta_{t^-}(\pi(k)) = a(k), \forall 1 \le k \le n, k \ne j, \eta_{t^-}(\pi(j)) = a(j) - 1) \\
&= \sum_{1\le i\le n} \frac{\mathbb{1}_{a_i \ge 1}}{1 + \sum_{\ell=1,\ell\ne i}^n \mathbb{1}_{a_\ell \ge 1}} \mathcal{P}(\eta_{t^-}(k) = a(k), \forall 1 \le k \le n, k \ne j, \eta_{t^-}(j) = a(j) - 1)
\end{aligned}$$

where the third line is obtained using the conditional probability of B_t^j, and the last one is a consequence of the exchangeability of $\eta_{t^-}^n$. The result follows since the equality also holds for $\pi = Id$. $\diamondsuit$

Lemma 4.4 *For any stopping time τ, any $\mathbb{N}$ valued $\mathcal{F}_\tau$-measurable random variable $\mathcal{X}$, if the random vector $\eta_\tau^{\mathcal{X}} = (\eta_\tau(1), ..., \eta_\tau(\mathcal{X}))$ is exchangeable, and τ' is the first time after τ of an arrow pointing to a level $\le \mathcal{X}$, a death or a mutation at a level $\le \mathcal{X}$, then conditionally upon the fact that τ' is the time of a compensatory mutation, the random vector $\eta_{\tau'}^{\mathcal{X}} = (\eta_{\tau'}(1), ..., \eta_{\tau'}(\mathcal{X}))$ is exchangeable.*

Proof : This proof is really similar to the previous one, except that the term before the $\mathcal{P}$ which was $\frac{\mathbb{1}_{a_j \ge 1}}{1+\sum_{\ell=1,\ell\ne j}^n \mathbb{1}_{a_\ell \ge 1}}$ is now $\frac{1}{n}$. $\diamondsuit$

Lemma 4.5 *For any stopping time τ, any $\mathbb{N}$ valued $\mathcal{F}_\tau$-measurable random variable $\mathcal{X}$, if the random vector $\eta_\tau^{\mathcal{X}} = (\eta_\tau(1), ..., \eta_\tau(\mathcal{X}))$ is exchangeable, and τ' is the first time after τ of an arrow pointing to a level $\le \mathcal{X}$, a death or a mutation at a level $\le \mathcal{X}$, then conditionally upon the fact that τ' is the time of a k-type death, the random vector $\eta_{\tau'}^{\mathcal{X}} = (\eta_{\tau'}(1), ..., \eta_{\tau'}(\mathcal{X} - 1))$ is exchangeable.*

Proof : To ease the notation we will condition upon $\mathcal{X} = n$ and $\tau' = t$, and denote $\mathcal{P}$ the associated conditional probability. Let a^{n-1} be a $n-1$ dimensional vector, and $\forall 1 \leq j \leq n$

$C_t^{j,k} = \{$ The k-th type death which occurs at time t involves the individual sitting on site j) $\}$.

$\forall \pi \in S_{n-1}$,

$$\mathcal{P}(\pi(\eta_t^{n-1}) = a^{n-1}) = \sum_{1\leq j\leq n-1} \mathcal{P}(\eta_t^{n-1} = \pi^{-1}(a^{n-1}), C_t^{j,k})$$

$$= \sum_{1\leq j\leq n} \mathcal{P}(\eta_t(1) = a_1^\pi, ..., \eta_t(n-1) = a_{n-1}^\pi, C_t^{j,k})$$

$$= \sum_{1\leq j\leq n} \mathcal{P}(\eta_{t^-}(1) = a_1^\pi, ..., \eta_{t^-}(j-1) = a_{j-1}^\pi, \eta_{t^-}(j) = k, \eta_{t^-}(j+1) = a_j^\pi, ...\eta_{t^-}(n) = a_{n-1}^\pi, C_t^{j,k})$$

$$= \frac{1}{1 + \sum_{1\leq \ell \leq n-1} \mathbb{1}_{a_\ell^\pi = k}} \sum_{1\leq j\leq n} \mathcal{P}(\eta_{t^-}(1) = a_1^\pi, ..., \eta_{t^-}(j-1) = a_{j-1}^\pi, \eta_{t^-}(j) = k,$$
$$\eta_{t^-}(j+1) = a_j^\pi, ...\eta_{t^-}(n) = a_{n-1}^\pi).$$
$$= \frac{1}{1 + \sum_{1\leq \ell \leq n-1} \mathbb{1}_{a_\ell = k}} \sum_{1\leq j\leq n} \mathcal{P}(\eta_{t^-}(1) = a_1, ...\eta_{t^-}(n-1) = a_{n-1}, \eta_{t^-}(n) = k).$$

where the last one is a consequence of the exchangeability of $\eta_{t^-}^n$ and $\sum_{1\leq \ell\leq n-1} \mathbb{1}_{a_\ell^\pi = k} = \sum_{1\leq \ell\leq n-1} \mathbb{1}_{a_\ell = k}$. The result follows. $\diamondsuit$

Proof of Proposition 4.1 : Let us define $\xi_t^n = \xi_t^{n,\infty}$. We have, $\forall T > 0$,

$$\inf_{0\leq t\leq T} \xi_t^n \to_{n\to\infty} \infty. \tag{4.1}$$

Indeed, $\forall N > 0$, $\forall T > 0$, with the same ideas as the proof of Proposition 3.1 with $\tilde{\rho}_t^n$ a birth and death process, starting from $\tilde{\rho}_0^n = n$, with a birth rate $\frac{cn(n-2)}{8}$), and a death rate $2n(M_1^{2n}(0)+4\sqrt{n})$, $\tau^n = \inf\{t>0, \tilde{\rho}_t^n = n/2\}$, we have :

$$\mathbb{P}\left(\left\{\inf_{0\le s\le T}\xi_s^N \le N/2\right\}\right) \le \mathbb{P}\left(\left\{\exists 0\le s_0 < s_1 \le T, \xi_{s_0}^N = N, \xi_{s_1}^N = N/2, \max_{s_0\le s\le s_1}\xi_s^N = N\right\}\right)$$

$$\le \mathbb{P}\left(\left\{\exists 0\le s\le T, \tilde{\rho}_s^N = N/2\right\}\bigcap\left\{\sup_{0\le r\le T} M_1^{2N}(r) - M_1^{2N}(0) \le 4\sqrt{N}\right\}\right.$$

$$\left.\bigcap\left\{\forall 1\le i\le N, 0\le t\le T\ \ \eta_i^{2N}(t) = \eta_i(t)\right\}\right)$$

$$+ p_{2N} + \sum_{k\ge N}\left(k\left(\frac{16\alpha k(M_1^{2k}(0)+4\sqrt{k})}{ck^2}\right)^{\frac{k}{2}} + p_{2k}\right)$$

$$\le \mathbb{P}\left(\left\{\tau^N < \infty\right\}\right) + p_{2N} + \sum_{k\ge N}\left(k\left(\frac{16\alpha k(M_1^{2k}(0)+5\sqrt{k})}{ck^2}\right)^{\frac{k}{2}} + p_{2k}\right)$$

$$\le p_{2N} + N\left(\frac{16\alpha N(M_1^{2N}(0)+5\sqrt{N})}{cN^2}\right)^{\frac{N}{2}} + \sum_{k\ge N}\left(k\left(\frac{16\alpha k(M_1^{2k}(0)+5\sqrt{k})}{ck^2}\right)^{\frac{k}{2}} + p_{2k}\right)$$

where the second line follows from Proposition 3.1 and from :

$$\left\{\exists 1\le i\le N, 0\le t\le T \text{ such that } \eta_i^{2N}(t)\neq\eta_i(t)\right\}$$
$$\subset \bigcup_{k\ge N}\left\{\exists 1\le i\le k, 0\le t\le T \text{ such that } \eta_i^{2k}(t)\neq\eta_i^{4k}(t)\right\},$$

since the rate at which ξ_s^N decreases on the set writen on the second line is

$$\alpha\sum_{k=0}^{\xi_s^N}\eta_k(s) \le \alpha\sum_{k=0}^{N}\eta_k(s) = \alpha\sum_{k=0}^{N}\eta_k^{2N}(s)$$
$$\le \alpha\sum_{k=0}^{2N}\eta_k^{2N}(s) = 2N\alpha M_1^{2N}(s)$$

Note that if we call q_N the upper-bound in the last line, we have $\sum_{N\ge1} q_N < \infty$, so (4.1) is a consequence of Borel-Cantelli Lemma . It follows from the previous Lemma that for each $t>0$, $n\ge 1$, $(\eta_t(1),...,\eta_t(\xi_t^{n,n}))$ is an exchangeable random vector. Consequently, for any $t>0$, $n'\ge 1$, $\pi\in S_n$, $a^{n'}\in\mathbb{N}^{n'}$,

$$|\mathbb{P}(\eta_t^{n'} = a^{n'}) - \mathbb{P}(\eta_t^{n'} = \pi^{-1}(a^{n'}))| \le \mathbb{P}(\xi_t^n \le n')$$

which goes to zero, as $n \to \infty$. The result follows. $\diamondsuit$

Theorem 2 results from Proposition 3.2 and Proposition 4.1.

5 A stronger convergence result

Now that we have defined the infinite model L^∞ and shown the convergence $\eta^{n,i} \Rightarrow \eta^{\infty,i}$, we can improve the convergence by using tightness, de Finetti theorem and some of our previous results. Note that a similar result appears in [5], Lemma A2.1. First, let us recall the theorem we will be using (see [1]).

Theorem 4 *An exchangeable (countably infinite) sequence $\{Y_n, n \geq 1\}$ of random variables is a mixture of i.i.d. sequences, in the sense that conditionally upon $\mathcal{G}$ (the tail $\sigma-$field of the sequence $\{Y_n, n \geq 1\}$) the Y_n s are i.i.d.*

From this one can deduce the following corollary (see corollary 1.6 in [3]) :

Corollary 5.1 $\forall k \geq 0$, $\forall T > 0$, $\forall t \in [0,T]$,

$$X_k^n(t) \to X_k(t) \ a.s. \ .$$

Then we can prove the following Proposition:

Proposition 5.2 $\forall k \geq 0$, $\forall \eta > 0$, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ *such as* $\forall n \geq n_0$ $\mathbb{P}(\sup_{0\leq t\leq T} |X_k^n(t) - X_k(t)| \geq \eta) \leq \varepsilon$

Proof :

We define $t_\ell = \ell \frac{\delta}{2}$ $\forall 0 \leq \ell \leq 2\frac{T}{\delta}$. Hence $\forall 0 \leq t \leq T$ $\exists \ell$ such as $|t - t_\ell| \leq \delta/2$. Then,

$$\begin{aligned}\sup_{0\leq t\leq T} |X_k^n(t) - X_k(t)| \leq & \sup_{0\leq \ell\leq 2\frac{T}{\delta}} \sup_{|r|\leq\delta/2} |X_k^n(t_\ell + r) - X_k^n(t_\ell)| + \sup_{0\leq \ell\leq 2\frac{T}{\delta}} |X_k^n(t_\ell) - X_k(t_\ell)| \\ & + \sup_{0\leq \ell\leq 2\frac{T}{\delta}} \sup_{|r|\leq\delta/2} |X_k(t_\ell + r) - X_k(t_\ell)|\end{aligned}$$

We fix $k \geq 0$, and let $\varepsilon, \eta > 0$ be arbitrary. The third term tends to 0 when δ tends to 0 since $X_k(t)$ is a continuous process, then

$$\exists \delta_0 > 0, \text{ such that } \forall \delta \leq \delta_0, \mathbb{P}\left(\sup_{0\leq \ell\leq 2\frac{T}{\delta}} \sup_{|r|\leq\delta/2} |X_k(t_\ell + r) - X_k(t_\ell)| \geq \frac{\eta}{3}\right) \leq \frac{\varepsilon}{3}.$$

For the first term, let us define, like in [4],

$$\omega_k^n(\delta) = \sup_{t,s\in[0,T]|t-s|\leq\delta} |X_k^n(t) - X_k^n(s)|$$

$$\omega_k'^n(\delta) = \inf_{(t_i,i\geq 0)\delta\text{-sparse}} \max_{i\geq 0} \sup_{t,s\in[t_i,t_{i+1}]} |X_k^n(t) - X_k^n(s)|.$$

Since the size of the jump of the process X^n are $1/n$, from (12.9) in [4] we have $\omega_k^n(\delta) \leq 2\omega_k'^n(\delta) + 1/n$. Moreover, since $(X_k^n, n \geq 0)$ is tight in $D([0,T])$, (see Theorem 13.2 in [4]) :

$$\exists\delta_1 \leq \delta_0, \forall\delta \leq \delta_1 \in \mathbb{N},\ \exists n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0, \mathbb{P}\left(\omega_k'^n(\delta) \geq \eta/6\right) \leq \varepsilon/6.$$

By combining those two results, we obtain that

$$\exists\delta_2 \leq \delta_0, \forall\delta \leq \delta_2\ \exists n_1 \in \mathbb{N} \text{ such that } \forall n \geq n_1, \forall k \in \mathbb{N}, \mathbb{P}\left(\omega_k^n(\delta) \geq \eta/3\right) \leq \varepsilon/3$$

which is true for the first term since :

$$\sup_{0\leq\ell\leq 2\frac{T}{\delta}} \sup_{|r|\leq\delta/2} |X_k^n(t_\ell + r) - X_k^n(t_\ell)| \leq \omega_k^n(\delta).$$

The second term tends to 0 as $n \to +\infty$ due to the previous Corollary, as $X_k^n(t_\ell)$ converge a.s. for each ℓ, and there are only a finite number of ℓ (which is a function of δ). Then,

$$\exists n_2 \geq n_1 \in \mathbb{N} \text{ such that } \forall n \geq n_2, \mathbb{P}\left(\sup_{0\leq\ell\leq 2\frac{T}{\delta}} |X_k^n(t_\ell) - X_k(t_\ell)| \geq \frac{\eta}{3}\right) \leq \frac{\varepsilon}{3}$$

So finally,

$$\exists n_2 \in \mathbb{N} \text{ such that } \forall n \geq n_2, \mathbb{P}\left(\sup_{0\leq t\leq T} |X_k^n(t) - X_k(t)| \geq \eta\right) \leq \varepsilon$$

$\diamondsuit$

Now with the Dini Theorem we can proceed to the :
PROOF OF THEOREM 3 : Let us define $S_k^n = \sum_{0\leq j\leq k} X_j^n$ and $S_k = S_k^\infty$. The S_k are increasing in k and continuous from $[0,T]$ in $\mathbb{R}^+$. Moreover, $\forall t \geq 0$ the $S_k(t) \to 1$

a.s. when $k \to \infty$.Then with the help of the first Dini Theorem, we obtain that the convergence is locally uniform. In other words,

$$\sup_{0\leq t\leq T} |1 - S_k(t)| \to_{k\to\infty} 0 \text{ a.s.}$$

Since S_k^n involves a finite number of X_ℓ^n, it converges uniformly in probability to S_k from Proposition 5.2, then $\forall \varepsilon > 0, \eta > 0$, $\exists K > 0$, $\exists n_0 > 0$ such as $\forall n \geq n_0$,

$$\mathbb{P}(\sup_{0\leq t\leq T} |1 - S_K(t)| \geq \eta/3) \leq \varepsilon/4.$$
$$\mathbb{P}(\sup_{0\leq t\leq T} \sum_{0\leq \ell\leq K} |X_\ell^n(t) - X_\ell(t)| \geq \eta/3) \leq \varepsilon/4.$$

.

Hence, noting that $|1 - S_k^n| \leq |1 - S_k| + |S_k - S_k^n|$,

$$\begin{aligned}
&\mathbb{P}(\sup_{0\leq t\leq T} \sum_{k\geq 0} |X_k^n(t) - X_k(t)| \geq \eta) \\
&\leq \mathbb{P}(\sup_{0\leq t\leq T} \sum_{0\leq \ell\leq K} |X_\ell^n(t) - X_l(t)| \geq \eta/3) + \mathbb{P}(\sup_{0\leq t\leq T} |1 - S_K| \geq \eta/3) + \mathbb{P}(\sup_{0\leq t\leq T} |1 - S_K^n| \geq \eta/3) \\
&\leq 2\mathbb{P}(\sup_{0\leq t\leq T} \sum_{0\leq \ell\leq K} |X_\ell^n(t) - X_l(t)| \geq \eta/3) + 2\mathbb{P}(\sup_{0\leq t\leq T} |1 - S_K| \geq \eta/3) \leq \varepsilon.
\end{aligned}$$

$\diamondsuit$

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